

# Algebraic Unsolvability of Problem of Absolute Stability of Desynchronized Systems Revisited

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**Abstract** In the author's article "Algebraic unsolvability of problem of absolute stability of desynchronized systems" (Automat. Remote Control 51 (1990), no. 6, pp. 754–759), it was shown that in general for linear desynchronized systems there are no algebraic criteria of absolute stability. In this paper, a few misprints occurred in the original version of the article are corrected, and two figures are added.

## 1 INTRODUCTION

In complex control systems containing sampled-data elements, it is possible that these elements operate asynchronously. In some cases asynchronous character of operation of sampled-data elements does not influence stability of system. In other cases any small desynchronization of the updating moments of sampled-data elements leads to dramatic changes of dynamics of a control system, and the system loses stability [1]. Last years there is begun (see, e.g., [1–5]) intensive studying of the effects connected with asynchronous operation of control systems; both necessary, and sufficient stability conditions for various classes of asynchronous systems were obtained. At the same time no one succeed in finding general, effectively verified criteria of stability of asynchronous systems, similar to known for synchronous systems [6]. The problem on stability of linear asynchronous systems has appeared more difficult than the problem on stability of synchronous systems. In the paper, attempt of formal explanation of complexity of the stability analysis problem for linear asynchronous systems is undertaken. It is shown that there are no criteria of absolute stability of linear asynchronous systems consisting of a finite number of arithmetic operations.

## 2 STATEMENT OF PROBLEM

Consider a discrete-time linear control system whose dynamics is described by the vector difference equation

$$x(n) = A(n)x(n-1) \quad (n = 1, 2, \dots), \quad (1)$$

where  $x(n) = \{x_1(n), x_2(n), \dots, x_N(n)\}$  is the state vector of the system and  $A(n) = (a_{ij}(n))$  is a square matrix of dimension  $N$  with the elements  $a_{ij}(n)$ .

The system (1) will be called synchronous if  $A(n) \equiv \text{const}$ . If  $A(n) \not\equiv \text{const}$ , and the set  $\{A(n) : n = 1, 2, \dots\}$  consists of finitely many elements  $A_1, A_2, \dots, A_M$ , then the system (1) will be called asynchronous or desynchronized.

Let  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  be a finite totality of square matrices of dimension  $N$ . The system (1) will be called absolutely stable with respect to the class of matrices  $\mathfrak{A}$  (cf. [7]) if there exists  $c = c(\mathfrak{A})$  such that for any sequence of matrices  $A(n) \in \mathfrak{A}$  the following estimates hold:

$$\|A(n)A(n-1) \cdots A(1)x\| \leq c\|x\| \quad (n = 1, 2, \dots). \quad (2)$$

Let us call the system (1) absolutely exponentially stable with respect to the class of matrices  $\mathfrak{A}$  if there exist  $c = c(\mathfrak{A})$  and  $q = q(\mathfrak{A}) < 1$  such that for any sequence of matrices  $A(n) \in \mathfrak{A}$  the following estimates hold:

$$\|A(n)A(n-1)\cdots A(1)x\| \leq cq^n\|x\| \quad (n = 1, 2, \dots). \quad (3)$$

If the class of matrices  $\mathfrak{A}$  consists of the square matrices  $A_1 = (a_{1ij})$ ,  $A_2 = (a_{2ij})$ ,  $\dots$ ,  $A_M = (a_{Mij})$  of dimension  $N$  then, for its description, it suffices to specify  $MN^2$  numbers:  $a_{111}$ ,  $a_{112}$ ,  $\dots$ ,  $a_{1NN}$ ,  $a_{211}$ ,  $a_{212}$ ,  $\dots$ ,  $a_{2NN}$ ,  $\dots$ ,  $a_{M11}$ ,  $a_{M12}$ ,  $\dots$ ,  $a_{MNN}$ . Therefore, each class  $\mathfrak{A}$  consisting of  $M$  square matrices of dimension  $N$  can be treated as a point in some space  $\mathfrak{M}(M, N) = R^{MN^2}$ . Denote by  $S(M, N)$  the set of those classes  $\mathfrak{A}$  in the space  $\mathfrak{M}(M, N)$  with respect to which the system (1) is absolutely stable. By  $E(M, N)$  we denote the set of those classes  $\mathfrak{A} \in \mathfrak{M}(M, N)$  with respect to which the system (1) is absolutely exponentially stable.

Now, the problem of studying the absolute stability of the system (1) can be reformulated as the problem of description of the sets  $S(M, N)$  and  $E(M, N)$ ; the simpler in some sense the structure of the sets  $S(M, N)$  or  $E(M, N)$  the easier to obtain a criterion of absolute stability or absolute exponential stability.

The sets  $S(1, N)$  and  $E(1, N)$  allow a simple description. Indeed, each class  $\mathfrak{A} \in \mathfrak{M}(1, N)$  consists of a single matrix. Therefore, we need to obtain conditions of stability or asymptotic stability of some difference equation  $x(n) = Ax(n-1)$ . The Routh–Hurwitz stability criterion [6] allows to represent these conditions as a finite system of polynomial inequalities including the elements  $a_{ij}$  of the matrix  $A$ . Verification of the obtained inequalities can be performed by a finite number of arithmetic operations over the elements of the matrix  $A$ . In other words, the question whether an arbitrary class  $\mathfrak{A} = \{A\}$  belongs to the set  $S(1, N)$  or  $E(1, N)$  may be resolved by a finite number of arithmetic operations.

Is it possible, for  $M \geq 2$ , by a finite number of arithmetic operations to resolve the question whether an arbitrary class  $\mathfrak{A} \in \mathfrak{M}(M, N)$  belongs to the set  $S(M, N)$  or  $E(M, N)$ ? The answer to this question will be given in the next section.

### 3 MAIN RESULT

Let  $u = \{u_1, u_2, \dots, u_L\}$  be an element of the coordinate space  $R^L$ . A finite sum  $p(u) = \sum p_{i_1 i_2 \dots i_L} u_1^{i_1} u_2^{i_2} \cdots u_L^{i_L}$  with numerical coefficients  $p_{i_1 i_2 \dots i_L}$  is called a polynomial in variable  $u \in R^L$ . A set  $U \subseteq R^L$  is said to have the *SA-property* [8] if there exists a finite number of polynomials  $p_1(u)$ ,  $\dots$ ,  $p_k(u)$ ,  $p_{k+1}(u)$ ,  $\dots$ ,  $p_{k+l}(u)$  such that  $U$  coincides with the set of elements  $u \in R^L$  satisfying the condition

$$p_1(u) > 0, \dots, p_k(u) > 0, \quad p_{k+1}(u) = \dots = p_{k+l}(u) = 0. \quad (4)$$

A set  $U$  is called semialgebraic [8] if it is a unity of a finite number of the sets possessing the *SA-property*.

**Theorem 1.** *Let  $M, N \geq 2$ . If a subset  $U$  of the space  $\mathfrak{M}(M, N)$  satisfies conditions  $E(M, N) \subseteq U \subseteq S(M, N)$  then it is not semialgebraic.*

The proof of the theorem is given in the Appendix.

Semialgebraicity of a set is equivalent to the existence of a criterion (consisting in verification of a finite number of the conditions of the form (4)) which allows by a finite number of arithmetic operations of addition, subtraction, multiplication and comparison of numbers to establish belonging of an element to a given set. As seen from Theorem 1, neither the set  $S(M, N)$  nor the set  $E(M, N)$  are semialgebraic. So, the meaning of Theorem 1 is that in general, by a finite number of arithmetic

operations, it is impossible to ascertain whether a desynchronized system (1) is absolutely stable (absolutely exponentially stable) or not.

The problem on the existence of algebraic criteria of stability is acute also for classes of desynchronized systems different from those considered above. For example, in the theory of continuous-time desynchronized systems there arises the problem of stability of the so-called regular systems [3, 4] (i.e., the systems with the infinite number of updating moments for each component). The discrete-time system (1), and the related to it sequence of matrices  $A(n)$ , will be called regular if each matrix  $A_i$  from the class  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  appears in the sequence  $\{A(n)\}$  infinitely many times. Denote by  $r(n)$  the greatest integer  $r$  having the property: the set of matrices  $\{A(1), A(2), \dots, A(n)\}$  can be decomposed in  $r$  subsets  $\{A(1), \dots, A(n_1)\}$ ,  $\{A(n_1 + 1), \dots, A(n_2)\}$ ,  $\dots$ ,  $\{A(n_{r-1} + 1), \dots, A(n)\}$  such that each of them contains all the matrices  $A_1, A_2, \dots, A_M$ . Clearly, the system (1) is regular if and only if  $r(n) \rightarrow \infty$ .

The system (1) will be called absolutely exponentially regularly stable with respect to the class of matrices  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  if there exist  $c = c(\mathfrak{A})$ ,  $q = q(\mathfrak{A}) < 1$  such that for any regular sequence of matrices  $A(n) \in \mathfrak{A}$  the following estimates hold:

$$\|A(n)A(n-1) \cdots A(1)x\| \leq cq^{r(n)}\|x\|.$$

Denote by  $R(M, N)$  the set all the classes  $\mathfrak{A} \in \mathfrak{M}(M, N)$  with respect to which the system (1) is absolutely exponentially regularly stable.

**Theorem 2.** *Let  $M, N \geq 2$ . Then the set  $R(M, N)$  is not semialgebraic.*

To prove Theorem 2 it suffices to note that the set  $R(M, N)$  contains  $E(M, N)$ , and is contained in  $S(M, N)$ . Then by Theorem 1 it is not semialgebraic. In other words, for  $M, N \geq 2$  there are no semialgebraic criteria of absolutely exponentially regular stability of discrete-time desynchronized systems.

#### 4 ADDENDUM

As shown above, the problem of absolute stability of the system (1) can be reduced to the analysis of behaviour of infinite products of the matrices  $A(n) \in \mathfrak{A}$ . Theorem 3 below reduces the same problem to the descriptive-geometric question on existence in the space  $R^N$  such a norm in which each matrix  $A_1, A_2, \dots, A_M$  is contractive.

**Theorem 3.** *The system (1) is absolutely stable in a class of matrices  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  if and only if there is a norm  $\|\cdot\|$  in  $R^N$  for which the following inequalities hold:*

$$\|A_1\|, \|A_2\|, \dots, \|A_M\| \leq 1. \quad (5)$$

*The system (1) is absolutely exponentially stable in a class of matrices  $\mathfrak{A}$  if and only if there is a norm  $\|\cdot\|$  in  $R^N$  and a number  $q < 1$  for which the following inequalities hold:*

$$\|A_1\|, \|A_2\|, \dots, \|A_M\| \leq q. \quad (6)$$

The proof of the theorem is given in the Appendix.

Several important properties of the sets  $S(M, N)$  and  $E(M, N)$  follow from Theorem 3. For example, the set  $E(M, N)$  is open in  $\mathfrak{M}(M, N)$ ; the set  $E(M, N)$  belongs to the interior of the set  $S(M, N)$ .

Due to openness of the set  $E(M, N)$ , if the system (1) is absolutely exponentially stable with respect to some class  $\mathfrak{A}^0 = \{A_1^0, A_2^0, \dots, A_M^0\}$  then it is also absolutely exponentially stable with respect to any class  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  of matrices  $A_i$  sufficiently close to the corresponding matrices  $A_i^0$  ( $i = 1, 2, \dots, M$ ).

Theorems 1 and 3 imply that the problem of construction, for a given set of square matrices, of a norm satisfying conditions (5) or (6) is algebraically unresolvable.

Theorem 1 states that in general there are no effective criteria of absolute stability of desynchronized systems (1). Nevertheless, such criteria may exist for some particular desynchronized systems. Let us present examples.

**Example 1.** Denote by  $\mathfrak{R}(N)$  the subset of the space  $\mathfrak{M}(N, N)$  consisting of the classes  $\mathfrak{A} = \{A_1, A_2, \dots, A_N\}$  of matrices  $A_i$  of the form

$$A_i = \begin{pmatrix} 1 & 0 & \dots & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ a_{i1} & a_{i2} & \dots & a_{ii} & \dots & a_{iN} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & \dots & 1 \end{pmatrix}. \quad (7)$$

The problem on absolute stability of the system (1) with respect to the classes  $\mathfrak{A} \in \mathfrak{R}(N)$  arises in [1–5] in the process of study of continuous-time systems with a special types of desynchronization of updating moments.

**Theorem 4.** *The system (1) is absolutely stable with respect to the class of matrices  $\mathfrak{A} = \{A_1, A_2\} \in \mathfrak{R}(2)$  of the form (7) if and only if one of the following system of relations holds:*

- a)  $a_{11} = 1, a_{12} = a_{21} = 0, a_{22} = 1$ ;
- b)  $a_{11} = 1, a_{12} = 0, a_{22} = -1, a_{21}$  is arbitrary;
- c)  $a_{11} = -1, a_{12}$  is arbitrary,  $a_{21} = 0, a_{22} = 1$ ;
- d)  $a_{11} = a_{22} = -1, 0 \leq a_{12}, a_{21} < 4$ ;
- e)  $|a_{11}| < 1, |a_{22}| < 1, -(1 - |a_{11}|)(1 - |a_{22}|) \leq a_{12}a_{21} \leq (1 - a_{11})(1 - a_{22})$ .

The proof of Theorem 4 is cumbersome and so is skipped. Let us point out that the criterion of absolute stability of the system (1) with respect to the classes of matrices from  $\mathfrak{R}(2)$ , given in Theorem 4, is semialgebraic.

**Example 2.** Denote by  $\mathfrak{R}_+(N)$  the subset of the space  $\mathfrak{M}(N, N)$  consisting of the classes  $\mathfrak{A} = \{A_1, A_2, \dots, A_N\}$  of matrices  $A_i$  of the form (7) for which  $a_{ij} > 0$ . The set  $S(N, N) \cap \mathfrak{R}_+(N)$  is semialgebraic. The criterion of absolute stability of the system (1) with respect to the classes of matrices  $\mathfrak{A}$  from  $\mathfrak{R}_+(N)$  consists in verification that the maximal eigenvalue of the matrix  $A = (a_{ij})$  does not exceed 1. This assertion is proved similarly to Theorem 2 from the first part of [1].

## APPENDIX

1. *Proof of Theorem 1* suffices to present for the case  $M = N = 2$ . The idea of proof is simple. We construct two families of matrices depending on the real parameter  $t \in [-1, 1]$ :

$$G(t) = (1 - t^4) \begin{pmatrix} 1 & -\frac{t}{\sqrt{1-t^2}} \\ 0 & 0 \end{pmatrix}, \quad H(t) = (1 - t^4) \begin{pmatrix} 1 - 2t^2 & -2t\sqrt{1-t^2} \\ 2t\sqrt{1-t^2} & 1 - 2t^2 \end{pmatrix}. \quad (\text{A.1})$$

The set  $W$  of all the classes  $\mathfrak{A}$  of the form  $\mathfrak{A} = \mathfrak{A}(t) = \{G(t), H(t)\}$  forms in the space  $\mathfrak{M}(2, 2)$  an algebraic set. Suppose that the set  $U$  is semialgebraic. Then the set  $U \cap W$  is also semialgebraic. Therefore, by the theorem of Whitney (see, e.g., [9]) about a finite number of connected components of a real algebraic set, a neighbourhood of the class  $\mathfrak{A}(0)$  in  $U \cap W$  should be either empty or

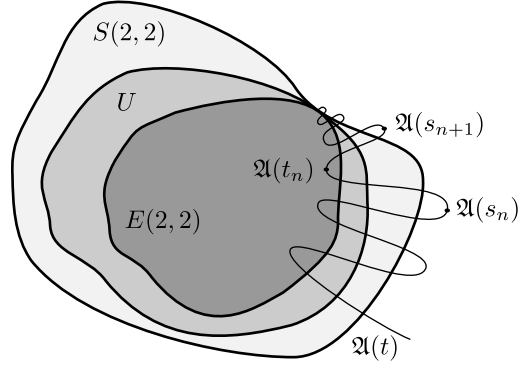


Figure 1: A case when the set  $U \cap W$  has infinitely many components of connectedness

consisting of a finite number of the connected components. We will show that it has infinitely many components of connectedness, see Fig. 1. So, the set  $U$  is not semialgebraic.

Let us pass to the theorem proof. Denote by  $|\cdot|$  the Euclidean norm in  $R^2$ : if  $x = \{\xi, \eta\}$  then  $|x| = \sqrt{\xi^2 + \eta^2}$ . Consider two families of matrices:

$$P(\varphi) = \begin{pmatrix} 1 & -\tan \varphi \\ 0 & 0 \end{pmatrix}, \quad R(\varphi) = \begin{pmatrix} \cos 2\varphi & -\sin 2\varphi \\ \sin 2\varphi & \cos 2\varphi \end{pmatrix}. \quad (\text{A.2})$$

**Lemma 1.** *Let  $\varphi = \pi/(2n+1)$ . Then  $(PR^n P)(\varphi) = -(\cos \varphi)^{-1}P(\varphi)$ .*

**Lemma 2.** *Let  $\varphi = \pi/(2n)$ . Then  $(PR^m P)(\varphi) = \lambda_{m,n}P(\varphi)$ , where  $|\lambda_{m+n,n}| = |\lambda_{m,n}|$ ,  $|\lambda_{m,n}| \leq 1$ .*

Both lemmas follow from the equality  $(PR^m P)(\varphi) = \frac{\cos[(2m+1)\varphi]}{\cos \varphi}P(\varphi)$  ( $m = 0, 1, \dots$ ) whose validity is justified by direct calculations, see Fig. 2.

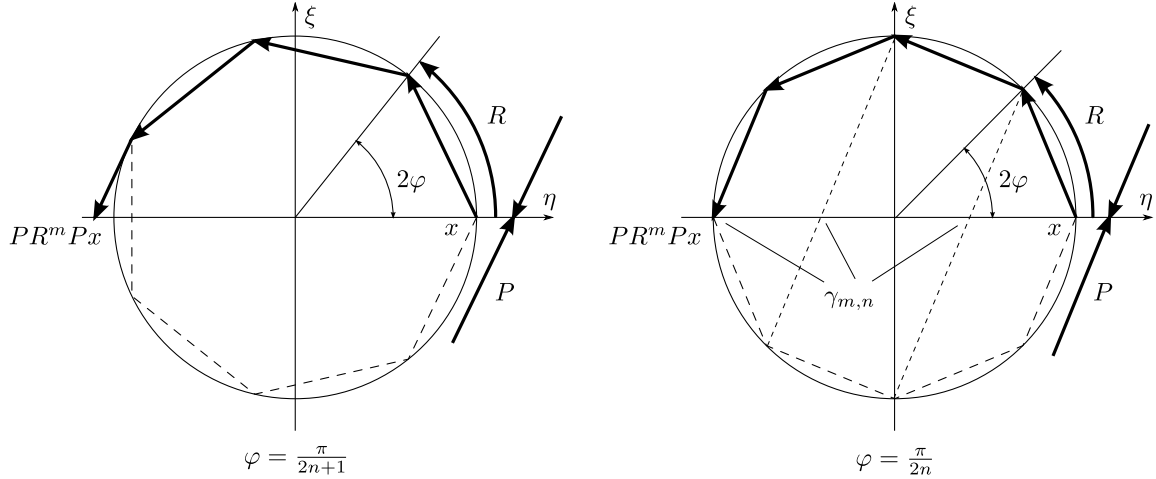


Figure 2: Iterations of a point  $x$  under the action of the map  $PR^m P$

**Lemma 3.** *Let  $\varphi = \pi/(2n)$ , and  $B_i = P(\varphi)$  or  $B_i = R(\varphi)$  for  $1 \leq i \leq m$ . Then*

$$B_m B_{m-1} \cdots B_1 = \alpha R^q(\varphi) P^r(\varphi) R^s(\varphi), \quad (\text{A.3})$$

where  $|\alpha| \leq 1$ , integers  $q$  and  $s$  are non-negative,  $r = 0, 1$ .

Prove the lemma by induction. For  $m = 1$  the assertion of the lemma is evident; suppose that it is valid for  $m = k - 1 \geq 1$ . Then for  $m = k$  the matrix  $A = B_m B_{m-1} \cdots B_1$  can be represented as  $A = B_m \tilde{\alpha} R^{\tilde{q}} P^{\tilde{r}} R^{\tilde{s}}$ , where  $|\tilde{\alpha}| \leq 1$ ,  $\tilde{r} = 0$  or  $\tilde{r} = 1$ ,  $R = R(\varphi)$ ,  $P = P(\varphi)$ .

If  $B_m = R(\varphi)$  then  $A = \tilde{\alpha} R^{\tilde{q}+1} P^{\tilde{r}} R^{\tilde{s}}$ , and for the matrix  $A$  the representation (A.3) holds in which  $\alpha = \tilde{\alpha}$ ,  $q = \tilde{q} + 1$ ,  $r = \tilde{r}$ ,  $s = \tilde{s}$ .

If  $B_m = P(\varphi)$  and  $\tilde{r} = 0$  then  $A = \tilde{\alpha} P R^{\tilde{q}+\tilde{s}}$ , and for the matrix  $A$  the representation (A.3) holds in which  $\alpha = \tilde{\alpha}$ ,  $q = 0$ ,  $r = 1$ ,  $s = \tilde{q} + \tilde{s}$ .

If  $B_m = P(\varphi)$  and  $\tilde{r} = 1$  then  $A = \tilde{\alpha} P R^{\tilde{q}} P R^{\tilde{s}}$ . Here the factor  $P R^{\tilde{q}} P$  according to Lemma 2 can be replaced by  $\lambda_{\tilde{q},n} P$ . Then  $A = \tilde{\alpha} \lambda_{\tilde{q},n} P R^{\tilde{s}}$ . Therefore, for the matrix  $A$  the representation (A.3) holds in which  $\alpha = \tilde{\alpha} \lambda_{\tilde{q},n}$ ,  $q = 0$ ,  $r = 1$ ,  $s = \tilde{s}$ . In addition,  $|\alpha| \leq |\tilde{\alpha}| \cdot |\lambda_{\tilde{q},n}| \leq 1$  since  $|\tilde{\alpha}| \leq 1$ ,  $|\lambda_{\tilde{q},n}| \leq 1$ .

The inductive step is completed. Lemma 3 is proved.

**Corollary.**  $|B_m B_{m-1} \cdots B_1| \leq |P(\frac{\pi}{2n})|$ .

The proof of the corollary immediates from the representation (A.3) and unitarity of the rotation matrix  $R(\varphi)$ .

**Lemma 4.** Let  $t_n = \sin \frac{\pi}{2n}$ . Then  $\mathfrak{A}(t_n) \in E(2, 2)$ .

Proof. Let  $\{A(k)\}$  be a sequence of matrices from  $\mathfrak{A}(t_n)$ . Then, for each  $k$ , one of two equalities  $A(k) = G(t_n)$  or  $A(k) = H(t_n)$  holds. By (A.1)  $G(t_n) = \mu_n P(\frac{\pi}{2n})$ ,  $H(t_n) = \mu_n R(\frac{\pi}{2n})$ , where  $\mu_n = 1 - (\sin \frac{\pi}{2n})^4$ . Therefore, the product of matrices  $A(1), A(2), \dots, A(k)$  can be represented in the form:  $A(k)A(k-1) \cdots A(1) = \mu_n^k B_k B_{k-1} \cdots B_1$ , where  $B_i = P(\frac{\pi}{2n})$  or  $B_i = R(\frac{\pi}{2n})$ . Then, by Corollary from Lemma 3,  $|A(k)A(k-1) \cdots A(1)| \leq \mu_n^k |P(\frac{\pi}{2n})|$  which implies absolute stability of the class of matrices  $\mathfrak{A}(t_n)$ . Lemma 4 is proved.

**Lemma 5.** Let  $s_n = \sin \frac{\pi}{2n+1}$ . Then  $\mathfrak{A}(s_n) \notin S(2, 2)$  for all sufficiently large  $n$ .

Proof. Clearly the lemma will be proved if, for each sufficiently large  $n$ , there can be found a sequence of matrices  $A(k) \in \mathfrak{F}(s_n)$  such that

$$|A(k_i)A(k_i-1) \cdots A(1)| \rightarrow \infty \quad (\text{A.4})$$

for some  $k_i \rightarrow \infty$ .

Define the sequence of matrices  $A(k)$  as follows:  $A[(n+2)i] = G(s_n)$ ,  $A[(n+2)i+1] = \dots = A[(n+2)i+n] = H(s_n)$ ,  $A[(n+2)i+n+1] = G(s_n)$ . Let us set  $k_i = (n+2)i+n+1$ . Then

$$A(k_i)A(k_i-1) \cdots A(1) = [G(s_n)H^n(s_n)G(s_n)]^i.$$

Since  $G(s_n) = \nu_n P(\frac{\pi}{2n+1})$  and  $H(s_n) = \nu_n Q(\frac{\pi}{2n+1})$ , where  $\nu_n = 1 - (\sin \frac{\pi}{2n+1})^4$ , then

$$A(k_i)A(k_i-1) \cdots A(1) = \left[ \nu_n^{n+2} P\left(\frac{\pi}{2n+1}\right) Q^n\left(\frac{\pi}{2n+1}\right) P\left(\frac{\pi}{2n+1}\right) \right]^i.$$

Consequently, by Lemma 1  $A(k_i)A(k_i-1) \cdots A(1) = \left( -\nu_n^{n+2} / \cos \frac{\pi}{2n+1} \right)^i P\left(\frac{\pi}{2n+1}\right)$ . Recall that  $P\left(\frac{\pi}{2n+1}\right)$  is a projector and so  $\left| P\left(\frac{\pi}{2n+1}\right) \right| \geq 1$ . Therefore,

$$|A(k_i)A(k_i-1) \cdots A(1)| \geq \left| -\frac{\nu_n^{n+2}}{\cos \frac{\pi}{2n+1}} \right|^i. \quad (\text{A.5})$$

Direct calculations show that  $\nu_n^{n+2}/\cos \frac{\pi}{2n+1} = 1 + \frac{\pi^2}{2(2n+1)^2} + o(n^{-2})$ . Henceforth, for sufficiently large values of  $n$  the inequality  $\nu_n^{n+2}/\cos \frac{\pi}{2n+1} > 1$  holds. From this and from (A.5) the relation (A.4) follows. Lemma 5 is proved.

Let us complete the proof of the theorem. Since by Lemmas 4 and 5  $\mathfrak{A}(t_n) \in E(2, 2)$ ,  $\mathfrak{A}(s_n) \notin S(2, 2)$  and  $E(M, N) \subseteq U \subseteq S(M, N)$  then  $\mathfrak{A}(t_n) \in U$ ,  $\mathfrak{A}(s_n) \notin U$ . But because of the points  $t_n$  and  $s_n$  interleave between each other then the set  $U \cap W$  contains infinitely many different components of connectedness (different classes  $\mathfrak{A}(t_n)$  belong to different components of connectedness). Therefore, the set  $U \cap W$  by the theorem of Whitney [9] is not semialgebraic. But since  $W$  is an algebraic set then the set  $U$  is not semialgebraic. Theorem 1 is proved.

2. *Proof of Theorem 3.* In one side the assertion of Theorem 3 is obvious: absolute stability and absolute exponential stability of the system (1) with respect to the class  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  immediately follow from inequalities (5) and (6).

Let us show that absolute exponential stability of the system (1) with respect to the class of matrices  $\mathfrak{A} = \{A_1, A_2, \dots, A_M\}$  implies (6). Let, for some  $q < 1$ , the relation (3) be valid. Set  $\varkappa_0(x) = |x|$ ,  $\varkappa_n(x) = q^{-n} \max |B_1 B_2 \dots B_n x|$  ( $n > 1$ ), where  $|\cdot|$  is the Euclidean norm on  $R^N$ , and the maximum is taken over all possible collections of the matrices  $B_1, B_2, \dots, B_n \in \mathfrak{A}$ . Define the norm  $\|\cdot\|$  as follows:  $\|x\| = \sup_{n \geq 0} \varkappa_n(x)$ .

The function  $\|x\|$  is semiadditive and due to (3) it satisfies the relations  $|x| \leq \|x\| \leq \max\{1, c\} |x|$ . Henceforth,  $\|x\| = 0$  if and only if  $x = 0$ . Other properties of a norm are obvious for  $\|\cdot\|$ .

Let us justify inequalities (6). Clearly, for any  $k = 1, 2, \dots, M$  and  $n = 0, 1, \dots$  the estimate  $\varkappa_n(A_k x) \leq q \varkappa_{n+1}(x)$  is valid. Therefore,

$$\|A_k x\| = \sup_{n \geq 0} \varkappa_n(A_k x) \leq q \sup_{n \geq 0} \varkappa_{n+1}(x) = \sup_{n \geq 1} \varkappa_n(x) \leq q \|x\|.$$

From here  $\|A_k\| \leq q$  for  $k = 1, 2, \dots, M$ . Inequalities (6) are proved.

Construction of the norm  $\|\cdot\|$  and the proof of inequalities (5) in the case of absolute stability of the system (1) are carried out similarly. Theorem 3 is proved.

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## POST SCRIPTUM

In the foregoing text, a few misprints in the proof of Theorem 1 occurred in the original journal version of the article [10] were corrected, and two figures were added. The improved text was included in the monograph [11]. Generalization of the presented results can be found in [12].